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# DERIVATION OF HIGHER APPROXIMATIONS IN THE PROBLEM OF SPECIAL FLOWS IN PLANE LAVAL NOZZLES

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The method of expansion in series in self-similar components is used for deriving solutions of gasdynamic equations for some off-design modes of flow in a plane nozzle, particularly in the case when local supersonic zones interlock at the nozzle axis (the Taylor limit flow). Analysis of expansions in self-similar solutions is carried out simultaneously in the stream and the hodograph planes. The first three terms of expansion are determined for the Taylor limit flow in a plane Laval nozzle. In that case the correction terms retain the property of double symmetry about the horizontal and vertical axes that pass through the nozzle center.

The possibility of realizing two flow patterns with nonanalytic velocity distribution along the Laval nozzle axis besides that provided by Meyer's solution [1, 2] which is analytic in the nozzle center neighborhood was shown in [3, 4]. The self-similarity indices $n = 2, 3, 5,$ and 11 ($xy^{-n}$ is the invariant of self-similar solution) correspond in the class of self-similar solutions of the approximate system of transonic equations to gas flows in Laval nozzles. The gasdynamic interpretation of the case of $n = 5$ in addition to its other values was proposed in [5] in the form of an asymptotic retardation flow in a supersonic diffuser with shock wave formation.

It was found in [6] that solutions which correspond to the above indices are algebraic and admit a convenient parametric representation, and it was shown that one of the solutions corresponds to the limit case in which local supersonic zones are locked on the nozzle axis. These solutions were thoroughly discussed in [7], where the existence of a wide class of asymmetric nozzles was established.

The problem of determination of higher approximations for various types of asymptotic flows in nozzles was formulated in [4] and its partial solution, which shows the transition from approximate transonic equations to the exact Chaplygin equations, was obtained. However the question of parameters of form, i.e. of constants which usually appear in expansions in self-similar components and provide information on the form of nozzle walls. A complete solution is known only for the Meyer type flow [8].

The indicated above values of self-similarity index are remarkable by that with them the passage of the general integral of ordinary differential equations through a singular point of the nodal type is analytic, a point that represents the limit characteristic and determines the existence of additional asymptotics [3, 4]. Analysis of higher approximations show that they may retain that property, and that the condition of flow symmetry about the nozzle axis is immaterial, as is the case of the zero approximation [7].

**1.** Plane potential flows of ideal perfect gas are defined by the system of equations

$$(a^2 - \varphi_x^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (a^2 - \varphi_y^2)\varphi_{yy} = 0 \tag{1.1}$$
$$(\varphi_x^2 + \varphi_y^2) / 2 + a^2 / (\gamma - 1) = (\gamma + 1) / [2 (\gamma - 1)]$$

where $x$ and $y$ are Cartesian coordinates reduced to dimensionless form relative to some characteristic length, $\varphi$ is the dimensionless velocity potential, $a$ is the local speed of sound relative to its critical value, and $\gamma$ is the ratio of specific heats.

We seek a solution of system (1.1) of the form

$$\varphi = x + q_0(\zeta)y^{3n-2} + q_1(\zeta)y^{k_1} + q_2(\zeta)y^{k_2} + \ldots \qquad (1.2)$$
$$\zeta = (\gamma + 1)^{-1/3}xy^{-n}$$

where $\zeta$ is the self-similar variable and $k_1, k_2, \ldots$ are exponents that form a non-decreasing sequence, with $k_1 > 3n - 2$.

Substituting (1.2) into (1.1) we obtain for $q_0$ a nonlinear equation and for $q_i$ ($i = 1, 2, \ldots$) a recurrent system of linear equations

$$(n^2\zeta^2 - q_0')q_0'' - n\zeta(5n - 5)q_0' + (3n - 2)(3n - 3)q_0 = 0 \qquad (1.3)$$

$$(n^2\zeta^2 - q_0')q_i'' - [q_0'' + (2k_i - n - 1)n\zeta]q_i' + k_i(k_i - 1)q_i = H_i \qquad (1.4)$$

where $H_i$ depend on $q_0, \ldots, q_{i-1}$ and on their derivatives.

The system (1.3), (1.4) has the singular point $\zeta_c$ defined by the relationship $n^2\zeta_c^2 - q_0'(\zeta_c) = 0$. The generalized parabola $\zeta = \zeta_c$ corresponds in the flow plane to the limit characteristic of the transonic approximation.

We define exponents $k_i$ by the condition that the general integrals of Eqs. (1.4) must be analytic along the limit characteristic. It is sufficient to consider the appropriate homogeneous equations, since the right-hand sides of (1.4) regularly depend on $q_0, \ldots, q_{i-1}$ and their derivatives, and the particular solution of (1.4) is analytic if all preceding solutions are analytic. The exponent for which the general integral of Eqs. (1.4) is analytic at point $\zeta_c$ will be called singular and denoted by $k_i^*$.

Let $Q_i$ ($i = 1, 2, \ldots$) be the solution of the homogeneous equation corresponding to (1.4). We expand functions $q_0$ and $Q_i$ in power series in the neighborhood of the limit characteristic, and obtain

$$q_0 = a_m(\zeta - \zeta_c)^m, \quad Q_i = b_{im}(\zeta - \zeta_c)^m \qquad (1.5)$$

where summation is carried out over integral nonnegative values of the recurrent index. The coefficients $a_m$ and $b_{im}$ are determined by substitution into related differential equations and can be represented in the recurrent form

$$a_0 = 5(n\zeta_c)^3 / (9n - 6), \quad a_1 = (n\zeta_c)^2, \quad a_2 = (n - 1)n\zeta_c / 2 \qquad (1.6)$$
$$a_m = A_m / [mn\zeta_c(-7n + 5 + mn + m)]$$
$$A_m = -a_{m-1}[n^2(m - 4)^2 + 5n(m - 4) + 6] + \Sigma_m$$
$$\Sigma_m = \frac{m}{2}\sum_{l=3}^{m-1} l(m + 2 - l)a_l a_{m+2-l}, \quad m = 3, 4, \ldots$$
$$b_{im} = B_{im} / [mn\zeta_c(-2k_i - n + 1 + mn + m)]$$
$$B_{im} = b_{i,m-1}[n(m - 1)(2k_i - mn + n - 1) - k_i(k_i - 1)] + \Sigma_{im}$$

$$\Sigma_{im} = m \sum_{l=3}^{m+1} l\,(m+2-l)\, a_l b_{i,\,m+2-l}, \quad m = 1, 2, \ldots$$

where $\Sigma_3$ and $\Sigma_{i1}$ are to be set equal zero, and $b_{i0}$ is an arbitrary constant.

The indices $n = 2, 3, 5,$ and $11$ are exceptional in the sense that coefficients $a_3, a_4, a_5,$ and $a_6$ can be taken respectively for these as arbitrary constants, since the numerator and the denominator in (1.6) vanish. If for an arbitrary $n$ only a particular integral of Eq. (1.3) is analytic on the limit characteristic, then for the indicated $n$ this general integral has that property [3,4].

The analyticity of the general integral $Q_i$ is equivalent to the arbitrariness of one of the coefficients $b_{im}$ $(m \geqslant 1)$ in (1.5) which occurs when the numerator and denominator $b_{im}$ (1.6) simultaneously vanish. It is possible to show by the appropriate analysis that the particular values of $k_i^*$ form for $n = 2, 3, 5,$ and $11$ the following sequence:

$$k_i^* = 3n - 2 + i\Delta, \quad \Delta = (n+1)/2 \tag{1.7}$$

where $i$ is a positive integer and the condition

$$i \notin \{i_l\}, \quad i_l = [5n - 7 + l\,(6n - 6)]/(n+1), \quad l = 0, 1, \ldots \tag{1.8}$$

is satisfied.

Note that for solving system (1.1) it is necessary to include in expansion (1.2) not only exponents (1.7) with condition (1.8) but, also, exponents of the form $k_{mj} = 3n - 2 + m\Delta + j\,(2n-2)$, where $m = 0, 1, \ldots$ and $j = 1, 2, \ldots$.

We denote by $q_{mj}$ the coefficient at the power of $y$ with exponent $k_{mj}$ for which we have Eq. (1.4). If $k_{mj}$ does not coincide with any $k_i^*$, then the representative $q_{mj}$ which is analytic on the limit characteristic is a particular integral of that equation.

The inequality $\Delta < (2n-2)$ is satisfied for the considered indices of self-similarity. The case of $n = 2$ is taken as known and is not considered here. We denote by $E$ the integral part of number $(2n-2)/\Delta$. The solution of Eqs. (1.1) can now be represented in the form of series in nondecreasing powers of $y$

$$\varphi = x + y^{3n-2}\left(\sum_{i=0}^{E} q_i y^{i\Delta} + q_{01} y^{2n-2} + \ldots\right) \tag{1.9}$$

The first correction to the solution of transonic equations was determined in [9], which makes it possible to write the particular integral for $m = 0$, $j = 1$ in the form

$$q_{01} = 5^{-1}(\gamma+1)^{-1/3}\{(\gamma + {}^5/_2)[(3n-2)q_0 - n\zeta q_0'] - (\gamma - {}^5/_2)q_0\}q_0' \tag{1.10}$$

Determination of the flow field is more conveniently carried out using the velocity vector components $u = \varphi_x$ and $v = \varphi_y$. In conformity with (1.9) the expansions for $u$ and $v$ are of the form

$$u = 1 + (\gamma + 1)^{-1/3} y^{2n-2} \Big( \sum_{i=0}^{E} f_i y^{i\Delta} + f_{01} y^{2n-2} + \ldots \Big) \qquad (1.11)$$

$$v = y^{3n-3} \Big( \sum_{i=0}^{E} g_i y^{i\Delta} + g_{01} y^{2n-2} + \ldots \Big)$$

$$f_i = q_i', \quad g_i = k_i^* q_i - n\zeta q_i', \quad i = 0, \ldots, E$$

$$f_{01} = q_{01}', \quad g_{01} = (5n - 4) q_{01} - n\zeta q_{01}'$$

**2.** Using functions $f_0(\zeta)$ and $g_0(\zeta)$ derived in [6] we determine representative velocity components $f_i$ and $g_i$ $(i = 1, \ldots, E)$ for $n = 3, 5$, and $11$. Direct integration of related equations (1.4) is difficult, hence we resort to the hodograph method. Let us consider the stream function $\psi$ which depends on the angle of inclination $\theta$ of velocity and on the variable $\eta$ introduced by Frankl [1]. That function satisfies the equation

$$\eta \psi_{\theta\theta} + \psi_{\eta\eta} + b(\eta) \psi_\eta = 0 \qquad (2.1)$$

where $b(\eta)$ is known function [10]. In the transonic approximation Eq. (2.1) is replaced by the Tricomi equation

$$\eta \psi_{\theta\theta} + \psi_{\eta\eta} = 0 \qquad (2.2)$$

We represent the solution of Eq. (2.1) in the form of expansion

$$\psi = \rho^\lambda \sum_{i=0}^{E} \psi_i(t) \rho^{i\delta} + \rho^{\lambda + 2/3} \psi_{01}(t) + \ldots \qquad (2.3)$$

$$\rho = (\theta^2 + {}^4/_9 \eta^3)^{1/2}, \quad t = \theta\rho^{-1}, \quad \lambda = (3n - 3)^{-1}, \quad \delta = \lambda\Delta$$

which must correspond to expansion (1.9).

Substituting (2.3) into (2.1), for the determination of coefficients $\psi_i$ we obtain the sequence of equations

$$(1 - t^2)\psi_i'' - {}^4/_3 t \psi_i' + (\lambda + i\delta)(\lambda + {}^1/_3 + i\delta)\psi_i = 0, \ i = 0, \ldots, E \qquad (2.4)$$

The general integral of this equation is expressed in terms of the hypergeometric functions

$$\psi_i = A_i F(-\lambda/2 - i\delta/2,\ \lambda/2 + {}^1/_6 + i\delta/2,\ {}^1/_2;\ t^2) + B_i t F(-\lambda/2 + {}^1/_2 - i\delta/2,\ \lambda/2 + {}^2/_3 + i\delta/2,\ {}^3/_2;\ t^2)$$

where $A_i$ and $B_i$ are arbitrary constants.

The analysis carried out with the use of Schwartz tables [6] shows that functions $\psi_i$ are algebraic for $\lambda = {}^1/_6, {}^1/_{12}, {}^1/_{30}$ $(i \notin \{i_l\})$. They can be determined using formulas of differentiation of hypergeometric functions, commencing with the known solutions for $\psi_0$ when $\lambda = {}^1/_6, {}^1/_{12}, {}^1/_{30}$. Function $\psi_{01}$ satisfies the nonhomogeneous equation whose general integral is not always algebraic and may contain logarithmic terms [4]. A particular solution of that equation was obtained by Falkovich [10]; it corresponds to correction (1.10).

The potential $\varphi$ can also be expanded in series

$$\varphi = \rho^{\lambda+1/3} \sum_{i=0}^{E} \varphi_i(t)\rho^{i\delta} + \rho^{\lambda+1}\varphi_{01}(t) + \ldots$$

The coefficients of $\varphi_i$ are expressed in terms of $\psi_i$ as follows:

$$(\lambda + 1/_3 + i\delta)\varphi_i = (^3/_2)^{1/3}(1 - t^2)^{2/3}\psi_i', \quad i = 0, \ldots, E \tag{2.5}$$

Note that coefficients $\psi_i$ ($i = 0, \ldots, E$) satisfy the homogeneous equations (2.4) and are actually determined by the Tricomi equation (2.2). Hence the relationships of the approximate transonic theory remain valid up to the order $\rho^{\lambda+E\delta}$, in particular $y \approx \psi$ and $x \approx \varphi$, i.e.

$$y = \rho^{\lambda} \sum_{i=0}^{E} \rho^{i\delta}\psi_i + \ldots, \qquad x = \rho^{\lambda+1/3} \sum_{i=0}^{E} \rho^{i\delta}\varphi_i + \ldots \tag{2.6}$$

We expand the module of the velocity vector $V = (u^2 + v^2)^{1/2}$ in powers of $\eta$ [2]

$$V = V(\eta) = 1 - (\gamma + 1)^{-1/3}\eta + O(\eta^2)$$

Expansions of velocity vector components can then be written as follows:

$$u = V\cos\theta = 1 - (\gamma + 1)^{-1/3}(^3/_2)^{2/3}(1 - t^2)^{1/3}\rho^{2/3} + O(\rho^{4/3}) \tag{2.7}$$
$$v = V\sin\theta = t\rho + O(\rho^{5/3})$$

On the other hand, the substitution of (2.6) into (1.11) yields similar expansions of $u$ and $v$ in powers of $\rho$, which contain intermediate terms of the kind $\rho^{i\delta}$. The coefficients at such powers must be set equal zero, as implied by (2.7). From this we obtain the relation between the velocity representatives $f_i$ and $g_i$ in terms of preceding terms and known coefficients $\psi_i$ and $\varphi_i$. For compactness related formulas we use the notation

$$X_i = \varphi_i\psi_0^{-n-i\Delta}, \quad x_i = X_i y^{n+i\Delta}, \quad u_i = f_i y^{2n-2+i\Delta}$$
$$Y_i = \psi_i\psi_0^{-1-i\Delta}, \quad y_i = Y_i y^{1+i\Delta}, \quad v_i = g_i y^{3n-3+i\Delta}$$

For higher approximations of velocity we then have the expressions

$$u_1 = -\Delta_{01}, \quad u_2 = -\Delta_{02} - \Delta_{11} - \Delta_{01}^2, \quad u_3 = -\Delta_{03} - \Delta_{12} - \tag{2.8}$$
$$\Delta_{21} - \Delta_{11}^2 - \Delta_{01}^3 - \Delta_{012}, \ldots$$
$$\Delta_{ij} = u_{ix}x_j + u_{iy}y_j, \quad \Delta_{ij}^2 = (^1/_2)u_{ixx}x_j^2 + u_{ixy}x_jy_j + (^1/_2)u_{iyy}y_j^2$$
$$\Delta_{ij}^3 = (^1/_6)(u_{ixxx}x_j^3 + u_{iyyy}y_j^3) + (^1/_2)(u_{ixxy}x_j^2y_j + u_{ixyy}x_jy_j^2)$$
$$\Delta_{ijk} = u_{ixx}x_jx_k + u_{ixy}(x_jy_k + x_ky_j) + u_{iyy}y_jy_k, \ \ldots$$

where subscripts $x$ and $y$ denote the related partial derivative.

Similar formulas hold for $v_i$.

3. We shall indicate the form of coefficients $X_i$ and $Y_i$ for $n = 3, 5$, and 11. Let $E_0$, $G_0$, $E_i$, and $D_i$ be arbitrary constants, and $s$ be a parameter which admits real values. For $n = 3$ we have $E = 2$. Then

$$X_1 = -(3\sqrt{3}/5)\, G_0 D_1 M_1 N_0^{-5}, \quad Y_1 = D_1 N_1 N_0^{-3}$$
$$\zeta = -G_0 M_0 N_0^{-3}$$
$$M_0 = E_0 s^3 + 1 + \sqrt{3}(s^2 - E_0, s), \quad N_0 = s + E_0$$
$$M_1 = -E_1 s^5 + 1 + 5(s^4 - E_1 s), \quad N_1 = s^3 + E_1 - \sqrt{3}(E_1 s^2 - s)$$

Coefficients $X_2$ and $Y_2$ must be set equal zero in conformity with condition (1.8). We use here only the particular solution (2.8).

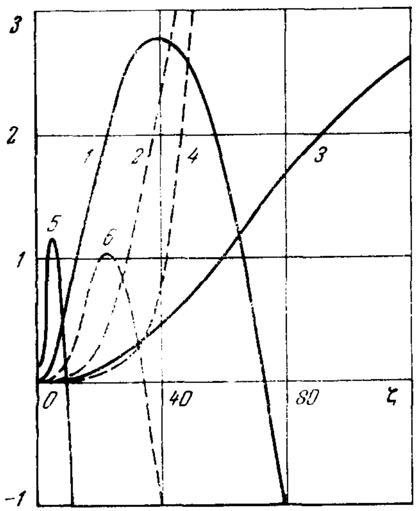


Fig. 1

Note that when $E_0 = E_1 = 0$ both, the principal terms and the higher approximations define a flow that is symmetric about the $x$- and the $y$-axes [6]. The dependence of $0.25\cdot10^{-2} f_1$, $0.25\cdot10^{-4} g_1$, $0.2\cdot10^{-4} f_2$, $0.2\cdot10^{-5} g_2$, $0.125\cdot10^{-2} f_{01}$, and $0.25\cdot10^{-5} g_{01}$ on the self similar variable for the following constants:

$$G_0 = (1/2)\sqrt{2+\sqrt{3}}, \quad D_1 = -5/(12\sqrt{3}G_0^2)$$

are shown in Fig. 1 by curves 1 – 6, respectively.

A class of asymmetric flows that obtain if at least one of constants $E_0$ or $E_1$ is nonzero is also possible.

For $n = 5$ we have $E = 2$, and then

$$X_1 = (5/2) G_0 D_1 M_1 N_0^{-8}, \quad Y_1 = D_1 N_1 N_0^{-4}$$
$$X_2 = (35/11)\, G_0 D_2 M_2 N_0^{-11}, \quad Y_2 = D_2 N_2 N_0^{-7}, \quad \zeta = G_0 M_0 N_0^{-5}$$
$$M_0 = -\sqrt{2}(s^5 + E_0) + 5(E_0 s^3 - s^2), \quad N_0 = E_0 s + 1$$
$$M_1 = -s^8 + E_1 + 8(E_1 s^6 - s^2) + 4\sqrt{2}(E_1 s^3 + s^5)$$
$$N_1 = E_1 s^4 + 1 + 2\sqrt{2}(s^3 - E_1 s)$$
$$M_2 = \sqrt{2}(s^{11} + E_2) + 11(E_2 s^9 - s^2) - 33(s^8 - E_2 s^3)$$
$$N_2 = E_2 s^7 + 1 - 7(s^6 + E_2 s) - 7\sqrt{2}(E_2 s^4 - s^3)$$

For $n = 11$ we obtain $E = 3$. The related representatives in the hodograph plane are defined as follows:

$$X_1 = (77/17)\, G_0 D_1 M_1 N_0^{-17}, \quad Y_1 = D_1 N_1 N_0^{-7}$$
$$X_2 = (143/23)\, G_0 D_2 M_2 N_0^{-23}, \quad Y_2 = D_2 N_2 N_0^{-13}$$
$$X_3 = (209/29)\, G_0 D_3 M_3 N_0^{-29}, \quad Y_3 = -D_3 N_3 N_0^{-19}$$
$$\zeta = G_0 M_0 N_0^{-11}$$
$$M_0 = s^{11} + E_0 - 11(E_0 s^{10} + s) + 66(s^6 - E_0 s^5), \quad N_0 = s + E_0$$

$$M_1 = E_1 s^{17} + 1 - 17 (s^{15} - E_1 s^2) + 119 (E_1 s^{12} - s^5) + 187 (s^{10} + E_1 s^7)$$
$$N_1 = E_1 s^7 + 1 + 7 (s^5 - E_1 s^2)$$
$$M_2 = s^{23} - E_2 - 46 (E_2 s^{20} - s^3) + 207 (s^{18} + E_2 s^5) + 1173 (E_2 s^{15} + s^8) - 391 (s^{13} - E_2 s^{10})$$
$$N_2 = s^{13} - E_2 + 26 (E_2 s^{10} - s^3) - 39 (s^8 + E_2 s^5)$$
$$M_3 = s^{29} - E_3 - 87 (E_3 s^{25} + s^4) + 435 (s^{24} + E_3 s^5) + 3335 (E_3 s^{20} - s^9) - 6670 (s^{19} - E_3 s^{10})$$
$$N_3 = -s^{19} + E_3 - 57 (E_3 s^{15} + s^4) + 171 (s^{14} + E_3 s^5) + 247 (E_3 s^{10} - s^9)$$

For the considered values of $n$ each coefficient $X_i$ and $Y_i$ is of the form

$$X_i = P_a N_0^{-a}, \quad Y_i = Q_b N_0^{-b}, \quad a = n + i\Delta, \; b = 1 + i\Delta$$

where $P_a$ and $Q_b$ are polynomials of corresponding power.

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